Problema 919. Let a, b, c be the lengths of the sides of triangle ABC with inradius r and radii of excircles r_a, r_b, r_c, respectively. Prove that

$$(1) \qquad (r_b - r_c)\cos A + (r_c - r_a)\cos B + (r_a - r_b)\cos C = 0 \text{ and}$$

$$(2) \qquad (r_b + r_c)\csc A + (r_c + r_a)\csc B + (r_a + r_b)\csc C = \frac{abc}{2r^2}.$$

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Let s, Δ be the semiperimeter and the area of $\triangle ABC$, respectively.

(1) By using the identities

$$r_a = s\tan\frac{A}{2} \quad , \quad r_b = s\tan\frac{B}{2} \quad , \quad r_c = s\tan\frac{C}{2}$$

and the sum-to-product formulas, we have:

$$\begin{aligned}
LHS &= \sum_{\text{cyclic}} r_a(\cos C - \cos B) = \sum_{\text{cyclic}} s\tan\frac{A}{2}(\cos C - \cos B) = \\
&= -\sum_{\text{cyclic}} 2s\tan\frac{A}{2}\sin\frac{B+C}{2}\sin\frac{B-C}{2} = -\sum_{\text{cyclic}} 2s\tan\frac{A}{2}\cos\frac{A}{2}\sin\frac{B-C}{2} = \\
&= -\sum_{\text{cyclic}} 2s\sin\frac{A}{2}\sin\frac{B-C}{2} = -\sum_{\text{cyclic}} 2s\cos\frac{B+C}{2}\sin\frac{B-C}{2} = \\
&= -\sum_{\text{cyclic}} s(\sin B - \sin C) = 0
\end{aligned}$$

(2) By using the identities

$$r_a = \frac{\Delta}{s-a} \quad , \quad r_b = \frac{\Delta}{s-b} \quad , \quad r_c = \frac{\Delta}{s-c}$$

and Heron's formula, we have:

$$\begin{aligned}
LHS &= \sum_{\text{cyclic}} \left(\frac{\Delta}{s-b} + \frac{\Delta}{s-c}\right)\frac{1}{\sin A} = \Delta\cdot\sum_{\text{cyclic}} \frac{2s-b-c}{(s-b)(s-c)}\cdot\frac{1}{\sin A} = \\
&= \Delta\cdot\sum_{\text{cyclic}} \frac{s(s-a)}{s(s-a)(s-b)(s-c)}\cdot\frac{a}{\sin A} = \Delta\frac{s}{\Delta^2}2R\sum_{\text{cyclic}}(s-a) = \\
&= \frac{s}{\Delta}\cdot 2\frac{abc}{4\Delta}\cdot s = \frac{s^2}{\Delta^2}\cdot\frac{abc}{2} = \frac{abc}{2r^2}
\end{aligned}$$

and the proof is complete. $\square$